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Several remarks on acceleration of convergence using generalized linear methods of summability

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Abstract

Some acceleration theorems for generalized linear summability methods $\mathcal{A} = (A_{nk})$, where A_{nk} are linear bounded operators from Banach space (B-space) X into B-space Y are derived. These results are proved by the method of summability with given rapidity. For this purpose some problems of summability with given rapidity are discussed.

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1. Preliminaries and notation

Let X and Y be B-spaces over the field \mathbf{K} , where $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$. Let $\mathcal{L}(X, Y)$ be a space of all bounded linear operators from X into Y . A convergent sequence $x = (\xi_k)$ with $\xi_k \in X$ and

$$\lim \xi_k = \xi \wedge \beta_k = \lambda_k(\xi_k - \xi)$$

is called λ -bounded (λ -convergent) if $\beta_k = O(1)$ ($\exists \lim \beta_k$), whereas $\lambda = (\lambda_k)$ with $0 < \lambda_k \nearrow$. Let m_X^λ and c_X^λ be, respectively, the sets of all λ -bounded and λ -convergent sequences $x = (\xi_k)$ with $\xi_k \in X$. In case $\lambda_k = O(1)$ we get $c_X^\lambda = m_X^\lambda = c_X$, where c_X is a set of convergent sequences $x = (\xi_k)$ with $\xi_k \in X$. Let m_X denote a set of bounded sequences while $x = (\xi_k)$ with $\xi_k \in X$ and $e_X(\zeta) := (\zeta, \zeta, \zeta, \dots)$ with $\zeta \in X$. A sequence $x = (\xi_k)$ is called (see [25,6]) summable by a generalized summability method $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, Y)$ or \mathcal{A} -summable if the sequence $y = (\eta_n)$ with

$$\eta_n = \sum_{k=0}^{\infty} A_{nk} \xi_k \tag{1}$$

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is convergent. The basic problems of generalized summability methods are studied in detail in [11,16–18]. By

$$\mathcal{A}m_X^\lambda \subset m_Y^\mu, \quad (2)$$

we denote that the transformation \mathcal{A} defined by (1) transforms every $x \in m_X^\lambda$ into $y \in m_Y^\mu$, while $\mu = (\mu_n)$ with $0 < \mu_n \nearrow$. The transformation \mathcal{A} is called accelerating λ -convergence or λ -boundedness correspondingly if $\mathcal{A}c_X^\lambda \subset c_Y^\mu$ or (2) with $\lim \mu_n/\lambda_n = \infty$ is valid.

In the case of number sequences and matrices the concepts of λ -convergence, λ -boundedness, λ -summability were first introduced in [7,8]. Kangro ([7,8], see also [10]) deduced necessary and sufficient conditions for $\mathcal{A}c^\lambda \subset c^\mu$ and $\mathcal{A}m^\lambda \subset m^\mu$, while $c^\lambda = c_{\mathbf{K}}^\lambda$ and $m^\lambda = m_{\mathbf{K}}^\lambda$ with $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$. Using the conditions for $\mathcal{A}c^\lambda \subset c^\mu$ Kangro [7] proved for number case that a regular triangular matrix transformation cannot accelerate λ -convergence. Using the properties of real numbers Kornfeld [9] proved in number case that any regular matrix method cannot accelerate the convergence. In the sequel, several Kangro's results are generalized for method $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, Y)$ where X and Y are B-spaces. The generalizations of Kangro's results from the setting of numerical sequences to the Banach space setting do not require essential modification if $X = Y$ and we use number matrices. In general case, we have to use the tools of functional analysis. Using our method we cannot generalize Kornfeld's result.

Lemma 1.1 (see [6]) is used to prove the main results.

Lemma 1.1. *Let X and Y be B-spaces, $A_{nk} \in \mathcal{L}(X, Y)$ and $\mathcal{A} = (A_{nk})$. If*

$$\exists \lim_n A_{nk} = A_k \quad (k \in \mathbf{N}_0)$$

in norm, then the conditions

$$\sum_k A_{nk} \xi_k \text{ is convergent for every } x \in m_X \quad (n \in \mathbf{N}_0)$$

and

$$\lim_n \sup_{\|\xi_k\| \leq 1} \left\| \sum_{k=0}^p (A_{nk} - A_k) \xi_k \right\| = 0 \quad (\text{uniformly } p \in \mathbf{N}_0)$$

are necessary and sufficient for $\mathcal{A}m_X \subset c_Y$. If these conditions are satisfied, then

$$x \in m_X \Rightarrow \lim_n \eta_n = \sum_k A_k \xi_k, \quad (3)$$

whereas $\sum_k A_k \xi_k$ is uniformly convergent with respect to $\|\xi_k\| \leq 1$.

2. Main results

Let in the sequel $\lambda_n \nearrow \infty$ and $\mu_n \nearrow \infty$.

Proposition 2.1. Let X and Y be B -spaces, $A_{nk} \in \mathcal{L}(X, Y)$ and $\mathcal{A} = (A_{nk})$. If

$$\exists \lim_n A_{nk} = A_k \quad (k \in \mathbb{N}_0) \quad \text{in norm,} \quad (4)$$

then the conditions

$$\mathcal{A}e_X(\zeta) \in m_Y^\mu \quad (\zeta \in X), \quad (5)$$

$$\sum_k \lambda_k^{-1} \|A_k\| < \infty, \quad (6)$$

$$\mu_n \sum_k \lambda_k^{-1} \|A_{nk} - A_k\| = O(1) \quad (7)$$

are necessary and sufficient for inclusion (2).

Proof. As $e_X(\zeta) \in m_X^\lambda (\zeta \in X)$, then condition (5) is necessary for (2). On the strength of condition (5) we have

$$\exists \lim_n \sum_k A_{nk} \zeta = A \zeta \quad (\zeta \in X). \quad (8)$$

Thereby, the quantity η_n defined by (1) may be presented in the form

$$\eta_n = \sum_k \lambda_k^{-1} A_{nk} \beta_k + \sum_k A_{nk} \zeta \quad (9)$$

with $\beta_k = \lambda_k (\zeta_k - \zeta)$. Taking into consideration condition (8) and presentation (9) we get $(\eta_n) \in c_Y \Leftrightarrow \mathcal{A}_\lambda(\beta_k) \in c_Y$ whereas $\mathcal{A}_\lambda := (\lambda_k^{-1} A_{nk})$. If

$$\exists \lim_n \lambda_k^{-1} A_{nk} = \lambda_k^{-1} A_k \quad (k \in \mathbb{N}_0) \quad \text{in norm,} \quad (10)$$

then according to Lemma 1.1 the necessary and sufficient conditions for inclusion $\mathcal{A}_\lambda m_X \subset c_Y$ are

$$\sum_k \lambda_k^{-1} A_{nk} \beta_k \quad (11)$$

is convergent $(\forall (\beta_k) \in m_X, n \in \mathbb{N}_0)$,

$$\lim_n \sup_{\|\xi_k\| \leq 1} \left\| \sum_{k=0}^p \lambda_k^{-1} (A_{nk} - A_k) \xi_k \right\| = 0 \quad (p \in \mathbb{N}_0) \quad (12)$$

uniformly with respect to p . Condition (11) we infer from

$$\sum_k \lambda_k^{-1} \|A_{nk}\| = O(1). \quad (13)$$

Condition (13) can be inferred from (4), (6) and (7). Vice versa (6) can be inferred from (4), (7) and (13). Condition (4) is equivalent to (10) and (12) can be inferred from (7). By (3), (9) and (8) we have

$$\eta = \lim_n \eta_n = \sum_k \lambda_k^{-1} A_k \beta_k + A \zeta.$$

As

$$(5) \Leftrightarrow \left((8) \wedge \mu_n \left(\sum_k A_{nk} \xi - A \xi \right) = O(1) \right),$$

we get

$$\begin{aligned} \mu_n(\eta_n - \eta) &= \mu_n \left(\sum_k \lambda_k^{-1} A_{nk} \beta_k + \sum_k A_{nk} \xi - \sum_k \lambda_k^{-1} A_k \beta_k - A \xi \right) \\ &= \mu_n \left(\sum_k \lambda_k^{-1} (A_{nk} - A_k) \beta_k \right) + \mu_n \left(\sum_k A_{nk} \xi - A \xi \right). \end{aligned}$$

Therefore,

$$(\eta_n) \in m_Y^\mu \Leftrightarrow \mathcal{A}_{\lambda, \mu}(\beta_k) \in m_Y$$

with $\mathcal{A}_{\lambda, \mu} = (\mu_n \lambda_k^{-1} (A_{nk} - A_k))$. Condition (7) is necessary and sufficient for inclusion $\mathcal{A}_{\lambda, \mu} m_X \subset m_Y$. If (4) is valid then conditions (5)–(7) are necessary and sufficient for inclusion (2). This completes the proof. \square

Let X be a B-space. A method $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, X)$ is called regular if $\mathcal{A}c_X \subset c_X$ and

$$\lim_n \eta_n = \lim_k \xi_k,$$

while $(\xi_k) \in c_X$ and η_n is defined by (1). As (see [18] or [24]) for regular \mathcal{A} we have $A_k = \theta$ and

$$\lim_n \sum_k A_{nk} = I,$$

where θ and I denote zero and identity operators, we infer from Proposition 2.1 the next result.

Corollary 2.1. *If $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, X)$ is a regular method and*

$$\sum_k A_{nk} = I \quad (n \in \mathbf{N}_0),$$

then the condition

$$\mu_n \sum_k \lambda_k^{-1} \|A_{nk}\| = O(1)$$

is necessary and sufficient for inclusion $\mathcal{A}m_X^\lambda \subset m_X^\mu$.

Remark 2.1. If we take $X = \mathbf{K}$ and $A_{nk} = a_{nk}I$ ($a_{nk} \in \mathbf{K}$) then we get from Proposition 2.1 Kangro's [8] corresponding theorem.

Remark 2.2. In case $\lambda_n \neq O(1)$ and $\mu_n = O(1)$ the assertion of Proposition 2.1 stays valid if the quantity $O(1)$ will be replaced in condition (7) with $o(1)$.

Remark 2.3. In [22] a similar result to Proposition 2.1 is proved for the inclusion $\mathcal{A}c_X^\lambda \subset c_Y^\mu$ using Zeller's [25] $c_X \rightarrow c_Y$ theorem instead of Kangro's [6] $m_X \rightarrow c_Y$ theorem.

Remark 2.4. Some examples to Proposition 2.1 can be derived using (see [3] or [12]) the lists of B-spaces and convenient linear bounded operators.

Example 2.1. Let $X = C[a, b]$ be B-space of real value continuous functions of a real variable, while the norm in $C[a, b]$ will be

$$\|\xi\| = \max_{a \leq t \leq b} |\xi(t)|.$$

Let the functions $K_{nk}(t, s)$ be continuous on $[a, b] \times [a, b]$. Then the integral operators $K_{nk} : C[a, b] \rightarrow C[a, b]$ defined by

$$(K_{nk}\xi)(t) = \int_a^b K_{nk}(t, s)\xi(s) \, ds \quad (t \in [a, b])$$

are linear and bounded (see [12]), while

$$\|K_{nk}\| = \max_{a \leq t \leq b} \int_a^b |K_{nk}(t, s)| \, ds.$$

If

$$\lim_n K_{nk} = K_k \quad (k \in \mathbb{N}_0) \quad \text{in norm}$$

and

$$\lim_n \sum_k K_{nk}\xi = K\xi \quad (\xi \in X),$$

then by Proposition 2.1 the conditions

$$\mu_n \max_{a \leq t \leq b} \left| \int_a^b \left(\sum_k K_{nk}(t, s) - K(t, s) \right) \, ds \right| = O(1),$$

$$\sum_k \lambda_k^{-1} \max_{a \leq t \leq b} \int_a^b |K_k(t, s)| \, ds < \infty,$$

$$\mu_n \sum_k \lambda_k^{-1} \max_{a \leq t \leq b} \int_a^b |K_{nk}(t, s) - K_k(t, s)| \, ds = O(1)$$

are necessary and sufficient for $\mathcal{H} m_{C[a, b]}^\lambda \subset m_{C[a, b]}^\mu$ with $\mathcal{H} = (K_{nk})$. If we have

$$K_k(t, s) = 0, \quad \sum_k K_{nk}(t, s) = K(t, s),$$

then the condition

$$\mu_n \sum_k \lambda_k^{-1} \max_{a \leq t \leq b} \int_a^b |K_{nk}(t, s)| \, ds = O(1)$$

is necessary and sufficient for $\mathcal{H} m_{C[a, b]}^\lambda \subset m_{C[a, b]}^\mu$. As this is a theoretical example we pose a problem on existing of a simple regular generalized summability method \mathcal{H} .

A generalized matrix transformation $\mathcal{A} = (A_{nk})$ is called triangular if $A_{nk} = \theta$ ($k > n$), where θ signifies the zero operator from X to Y .

Proposition 2.2. *Let X and Y be B -spaces and $A_{nk} \in \mathcal{L}(X, Y)$. If the triangular transformation $\mathcal{A} = (A_{nk})$ defined by*

$$\eta_n = \sum_{k=0}^n A_{nk} \xi_k \quad (14)$$

is accelerating λ -boundedness, then

$$\lim_n \sum_{k=0}^n \|A_{nk} - A_k\| = 0. \quad (15)$$

Proof. Suppose by contradiction that (15) is false. That means there exist $\varepsilon_0 > 0$ and indices v_n with

$$\sum_{k=0}^{v_n} \|A_{v_n k} - A_k\| \geq \varepsilon_0.$$

While $0 < \lambda_k \nearrow$, we get

$$\sum_{k=0}^{v_n} \lambda_k^{-1} \|A_{v_n k} - A_k\| \geq \varepsilon_0 \lambda_{v_n}^{-1}. \quad (16)$$

As a triangular transformation \mathcal{A} is accelerating λ -boundedness, then according to the notion of λ -boundedness acceleration and Proposition 2.1 we infer from condition (7) that

$$\mu_{v_n} \sum_{k=0}^{v_n} \lambda_k^{-1} \|A_{v_n k} - A_k\| = O(1). \quad (17)$$

Therefore by (16) and (17) we have

$$\varepsilon_0 \mu_{v_n} \lambda_{v_n}^{-1} \leq \mu_{v_n} \sum_{k=0}^{v_n} \lambda_k^{-1} \|A_{v_n k} - A_k\| = O(1).$$

So we get

$$\varepsilon_0 \mu_{v_n} \lambda_{v_n}^{-1} = O(1)$$

contradicting to the condition of acceleration $\lim \mu_n \lambda_n^{-1} = \infty$. So (15) is valid. This completes the proof. \square

Corollary 2.2. *If X is B -space and $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, X)$ is a regular triangular matrix method satisfying the condition*

$$\sum_{k=0}^n A_{nk} = I, \quad (18)$$

then \mathcal{A} cannot accelerate λ -boundedness.

Proof. By regularity of \mathcal{A} we have $A_k = \theta$ and, therefore, condition (15) is in the form

$$\lim_n \sum_{k=0}^n \|A_{nk}\| = 0$$

contradicting to condition (18). \square

Remark 2.5. In recent years, the main results in convergence acceleration have been proved for nonlinear methods of acceleration (see [4,5,13–15,23]). Though nonlinear methods usually give better results than linear methods and a regular triangular method \mathcal{A} satisfying conditions of Corollary 2.2 cannot accelerate λ -boundedness, regular triangular methods are used (see [21], also [2]) for acceleration of some subsets of m_X^λ . Also for acceleration in some cases there is a possibility to use conull methods. Conull methods satisfy condition (15).

In some cases, we have information about the transformed sequence $\mathcal{A}x$, namely

$$\mathcal{A}x \in m_X^\lambda. \quad (19)$$

Using additional conditions we want to infer from (19) the information about the rapidity of convergence of x . In some cases, we can use Tauberian remainder theorems. Let us present the simplest result of this type, while information about more complicated Tauberian remainder theorems can be found in [19,20].

Proposition 2.3. *Let X be B -space and $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, X)$ be a regular triangular method satisfying the conditions*

$$\begin{aligned} \lambda_n \sum_{k=0}^n \lambda_k^{-1} \|A_{nk}\| &= O(1), \\ \lambda_k \left\| \sum_{v=0}^k A_{nv} \right\| \|\Delta \xi_k\| &= O(\|A_{nk}\|) \quad (k \leq n, \quad n \in \mathbf{N}_0), \\ \lambda_n \left\| \sum_{k=0}^n A_{nk} - I \right\| \|\xi_n\| &= O(1), \end{aligned}$$

whereas $\Delta \xi_k = \xi_k - \xi_{k+1}$. Then $\mathcal{A}x \in m_X^\lambda$ implies $x \in m_X^\lambda$.

Proof. Let a sequence (η_n) be defined by (14). As

$$\begin{aligned} \lambda_n \|\xi_n - \eta\| &= \lambda_n \|(\xi_n - \eta_n) - (\eta_n - \eta)\| \\ &\leq \lambda_n \|\xi_n - \eta_n\| + \lambda_n \|\eta_n - \eta\|, \end{aligned}$$

$$\mathcal{A}x \in m_X^\lambda \Leftrightarrow \lambda_n \|\eta_n - \eta\| = O(1)$$

and using (see [1]) Abel's transformation we get

$$\begin{aligned}
 & \lambda_n \|\eta_n - \xi_n\| \\
 &= \lambda_n \left\| \sum_{k=0}^n A_{nk} \xi_k - \xi_n \right\| \\
 &= \lambda_n \left\| \sum_{k=0}^{n-1} \left(\sum_{v=0}^k A_{nv} \right) (\Delta \xi_k) + \left(\sum_{k=0}^n A_{nk} \right) \xi_n - \xi_n \right\| \\
 &\leq \lambda_n \sum_{k=0}^n \left\| \sum_{v=0}^k A_{nv} \right\| \|\Delta \xi_k\| + \lambda_n \left\| \sum_{k=0}^n A_{nk} - I \right\| \|\xi_n\| \\
 &= O \left(\lambda_n \sum_{k=0}^n \lambda_k^{-1} \|A_{nk}\| \right) + O(1) = O(1),
 \end{aligned}$$

then

$$\lambda_n \|\xi_n - \eta\| = O(1).$$

As \mathcal{A} is regular, then $\eta = \xi$. This completes the proof. \square

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